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Geometrization of local Langlands.

AKTS.

Let p be a prime. E NA local field. Residue field $\mathbb{F}_q$, $q = p^f$.

π fixed uniformizer.

$$\text{Either} \quad E = \mathbb{F}_q((\pi)) \quad \text{or} \quad [E : \mathbb{Q}_p] < \infty.$$

Fix $l \neq p$. All reps are on $\overline{\mathbb{Q}}_l$-v.s.

Local Langlands (for $G = GL_n$). Due to Harris–Taylor, Henniart.

$$\left\{ \begin{array}{c} \text{irred. smooth admissible} \\ \text{reps of } GL_n(E) \end{array} \right\} \Big/ \sim \quad \xleftrightarrow{1:1} \quad \left\{ \begin{array}{c} l\text{-adic reps of} \\ \underset{\text{Weil group}}{W_E} \text{ of dim } n \end{array} \right\}.$$

Bijection should have some nice properties, which uniquely characterize it.

<u>Recall</u>. (a) (π, V) rep. of $GL_n(E)$ is

* <u>smooth</u> if for all $v \in V$, stabilizer is open, and
* <u>admissible</u> for $K \subseteq GL_n(E)$ compact open (e.g. $GL_n(\mathcal{O}_E)$), $\dim_{\overline{\mathbb{Q}}_l} (V^K) < \infty$.

(b)
$$\begin{tikzcd}
 & & & \hat{\mathbb{Z}} \arrow[d, phantom, "\wr"] & \\
1 \arrow[r] & \overset{\text{inertia}}{I_E} \arrow[r] & \mathrm{Gal}(\bar{E}/E) \arrow[r] & \mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q) \arrow[r] & 1 \\
1 \arrow[r] & I_E \arrow[r] \arrow[u, equal] & W_E \arrow[r] \arrow[u, "\text{profinite completion}"'] & \mathbb{Z} \arrow[r] \arrow[u] & 1
\end{tikzcd}$$

<u>Fargues</u>. Replace this picture with

$$\begin{tikzcd}
\{GL_n(E)\}/\sim \arrow[d, squiggly] & & \{W_E\} \arrow[ll] \arrow[d, squiggly] \\
\left\{\begin{array}{c}\text{sheaves on the}\\ \text{stack of rank } n \text{ v.b.}\\ \text{on } X_E\end{array}\right\} & X_E \ \underset{\text{curve for } E}{\text{FF}} & \left\{\begin{array}{c}\ell\text{-adic local systems}\\ \text{on } X_E\end{array}\right\}
\end{tikzcd}$$

FF curve has two incarnations.

- Noetherian scheme over $\mathrm{Spec}(E)$ of dim 1.
- Adic space over $\mathrm{Spa}(E)$.

Not of f.t.

$\Big\}$ "X_E".

Definition is by uniformization.

$$X_E = Y_E/\varphi.$$

Puncture unit disks.

$\left(\text{If } E = \mathbb{F}_q((\pi)),\ Y_E = \mathbb{D}^*_{\mathbb{C}_p^\flat} \to \mathbb{D}^*_{\mathbb{F}_q} = \mathrm{Spa}(E) \right)$

$\mathcal{O}(Y_E) = \Big\{ \sum_{n\in\mathbb{Z}} x_n \pi^n,\ x_n \in \mathbb{C}_p^\flat,$

$\lim_{|n|\to\infty} |x_n| r^n \to 0,$

$\forall\, 0 < r < 1.$

Let b be an isocrystal over $\overline{\mathbb{F}}_p$.

$\quad\ \ \shortparallel$

(D, φ), D f. dim v.s. over $W(\overline{\mathbb{F}}_p)[\frac{1}{p}]$

$\varphi : D \xrightarrow{\sim} D$ semilinear.

Dieudonné–Manin

Category semisimple abelian with a simple object of slope $\lambda \in \mathbb{Q}$.

$\mathcal{E}_b$ vector bundle on X_E.

$$Y_E \times_\varphi D \to Y_E/\varphi = X_E.$$

$b \mapsto \mathcal{E}_b$ gives all vector bundles on the curve.
Also faithful, but not full.

$$\lambda \mapsto \mathcal{O}(\lambda)$$

<u>Thm</u> (FF). Any v.b. $\mathcal{E}$ on X_E decomposes as

on $\displaystyle\bigoplus_{\lambda\in\mathbb{Q}} \mathcal{O}(\lambda)^{n_\lambda}$.

Analogue of Grothendieck's theorem.

Computations. $H^0(X_E, \mathcal{O}(\lambda)) = 0, \quad \lambda < 0,$

$$H^1(X_E, \mathcal{O}(\lambda)) = 0, \quad \lambda \geq 0,$$

$$H^0(X_E, \mathcal{O}) = E,$$

$$H^0(X_E, \mathcal{O}(1)) = (B^+_{crys})^{\varphi = p}.$$

$$H^1(X_E, \mathcal{O}(-1)) = \mathbb{C}_p / E.$$

So, really not of finite type.

Cor. $\pi_1^{ét}(X_F) = \mathrm{Gal}_E.$

proof. Want $A \longmapsto \mathcal{O}_{X_E} \otimes_E A$

$$\left\{ \begin{array}{c} \text{finite étale} \\ E\text{-alg} \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \text{finite étale } \mathcal{O}_E \\ \text{algebras} \end{array} \right\}.$$

Let $\mathcal{E}$ be a finite étale $\mathcal{O}_E$-algebra, so $\mathcal{E} \underset{v.b.}{\simeq} \bigoplus \mathcal{O}(\lambda_i)$.
Trace pairing $\Rightarrow$ $\mathcal{E}$ is self-dual, so $\sum \lambda_i = 0$.
Pick λ maximal. $m: \mathcal{E} \otimes_{\mathcal{O}} \mathcal{E} \to \mathcal{E}$. Restrict to $\mathcal{O}(\lambda) \otimes \mathcal{O}(\lambda)$.
Since λ is maximal, if $\lambda > 0$, $m|_p$ is zero. But $\mathcal{E}$ is reduced.

Rem. Same proof works for $\mathbb{P}^1$.

Cor. $\{\ell\text{-adic rep of } G_E\} \leftrightarrow \{\ell\text{-adic local systems on } X_E\}.$

$$\{\ell\text{-adic reps of } W_E\}$$

more subtle, can be done geometrically.

Bun_n

Sheaves on the stack of rank n vbs on X_E.

Could try just using $X_E \to \mathrm{Spec}\, E$ or $\mathrm{Spa}\, E$.

But, not good.

$X_E = Y_E/\varphi$

$Y_E = \mathrm{Spa}\, E \times \mathrm{Spa}\, \mathbb{C}_p^\flat$

Replace $\mathbb{C}_p^\flat$ by some perfectoid space.

Instead go to sheaves or stacks on the cat. of all perfectoid spaces over $\overline{\mathbb{F}}_p$ with the pro-étale topology.

For $S \in \mathrm{Perf}_{\overline{\mathbb{F}}_p}$, $X_{S,E} \text{ "=" } (\mathrm{Spa}\, E \times S)/\varphi$. Now, S varies.

X_E still lives over $\mathrm{Spa}\, E$,

but no map $X_{S,E}$ to S.

<u>Def.</u> $\mathrm{Bun}_n(S)$ = groupoid of rank n vbs on $X_{S,E}$.

or $\mathrm{Bun}_{n,E}$.

<u>Thm</u> (Kedlaya–Liu). Actually a stack for the pro-étale topology.

<u>Rem.</u> (1) $\overset{\text{top. space}}{|\mathrm{Bun}_n|} = B(GL_n)$

rank n isocrystals.

$U \subseteq |\mathrm{Bun}_n|$ is open if $b \in U$, b' another isocrystal s.t. Newton polygon of b' is above b, then $b' \in U$.

<u>Cor.</u> $\mathrm{Bun}_n^{ss} \hookrightarrow \mathrm{Bun}_n$ is open.

(2) $\pi_0(\mathrm{Bun}_n) = \mathbb{Z}$ (degree).

(3) $b \in B(GL_n)$, $\widetilde{J}_b$ sheaf of aut of $\mathcal{E}_b$.

$\underline{J_b(E)}(S) = \mathrm{Hom}_{Top}(|S|, J_b(E))$.

Locally profinite.

$J_b(E) = \underline{J_b(E)}$ elts in $GL_n(\hat{E}^{un})$ $g b \sigma(g)^{-1} = b$.

$b \in B(GL_n)$

(4) $\mathrm{Bun}_n^b = [\mathrm{Spa}(\overline{\mathbb{F}}_p) / \cancel{J_b}]$,

Kedlaya–Liu.

(5) $\mathrm{Bun}_n^{ss} = \coprod_{b \text{ isoclinic}} [\cdot / \underline{J_b(E)}]$.

<u>Ex</u>. $b \longleftrightarrow \mathcal{E}_b = \mathcal{O}^n$

$J_b(E) = GL_n(E)$

$b \longleftrightarrow \mathcal{E}_b = \mathcal{O}(1/n)$

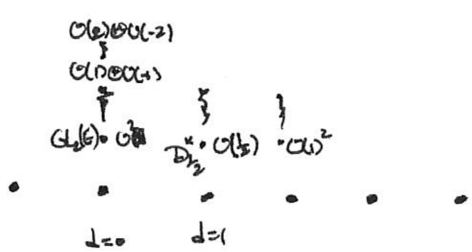
$J_b(E) = D_{1/n}^\times$

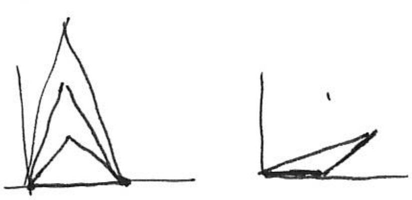
division alg. of inv. $1/n$ over E.

<u>Ex</u>. <u>$n=1$</u>. $\mathrm{Bun}_1 = \mathrm{Pic} = \coprod_{d \in \mathbb{Z}} \mathrm{Pic}^d$.

$\mathrm{Pic}^d = [\mathrm{Spa}(\overline{\mathbb{F}}_p) / \underline{E^\times}]$.

<u>Ex</u>. <u>$n=2$</u>. $\mathbb{Z}$ connected componts





ss. point always has <u>$d=0$</u>

G locally profinite group.

Prop. $[\mathrm{Spa}(\overline{\mathbb{F}}_p)/\underline{G}]$ is a smooth Artin stack.

$$D_{\acute{e}t}([*/G], \Lambda) \simeq D(\text{sm. rep. of } G \text{ on } \Lambda\text{-mod})$$

$\Lambda = \mathbb{Z}/\ell^n$, $n \geq 1$

Verdier duality $\longleftrightarrow$ smooth duality.

Cheat to get to $\Lambda = \overline{\mathbb{Q}}_\ell$.

$\mathcal{F} \in D_{\acute{e}t}([\cdot/\underline{G}], \Lambda)$ is reflexive ($\mathcal{F} \xrightarrow{\sim} \mathcal{F}^{\vee\vee}$)

$\iff$ for all $i \in \mathbb{Z}$,

$H^i(\mathcal{F})$ is <u>admissible</u>.

Thm (Fargues–Scholze). Bun_n is a smooth Artin stack.

$\mathcal{F} \in D_{\acute{e}t}(\mathrm{Bun}_n, \Lambda)$ is reflexive

$\Updownarrow$

$\forall\, b \in B(GL_n)$, $i_b^* \mathcal{F}$ is reflexive.

$i_b : \mathrm{Bun}_n^b \hookrightarrow \mathrm{Bun}_n$.

$$i_b^*\mathcal{F} \in D_{\acute{e}t}(\mathrm{Bun}_n^b) = D_{\acute{e}t}([\cdot/\widetilde{J}_b], \Lambda) = D_{\acute{e}t}([\cdot/J_b(E)], \Lambda).$$

Or, $H^j(i_b^*\mathcal{F})$ is admissible for all j, b.

Case of $n=1$. Local class field theory.

K = function field of sm. proj. curve over $\mathbb{F}_q$.

$\mathbb{A}$ ring of adèles of K.
$\quad \| $
$\prod' K_x^\wedge$, $\mathcal{O} = \prod_{x\in|X|} \mathcal{O}_x$

idèles

Thm. Unramified CFT, $G_K^{unr,ab} \simeq \left(K^\times \backslash \mathbb{A}^\times / \mathcal{O}^\times\right)^\wedge$.

$$\prod_{x\in|X|} \mathrm{Frob}_x^{\mathrm{ord}_x(a_x)} \longleftarrow (a_x).$$

$\boxed{\begin{array}{l} * \ G_K^{unr} = \pi_1(X), \\ * \ (\text{Weil}) \ K^\times\backslash\mathbb{A}^\times/\mathcal{O}^\times = \mathrm{Pic}_X(\mathbb{F}_q). \end{array}}$

Deligne's geometric proof.

$\uparrow$ sheaf-functions dictionary

$$\left\{\begin{array}{c}\text{character sheaves}\\ \text{on } \mathrm{Pic}_X\end{array}\right\} \xrightarrow[\substack{\text{Abel-Jacobi} \\ X \to \mathrm{Pic}^1_X \\ x \mapsto \mathcal{O}(x)}]{\sim} \left\{\begin{array}{c}\text{rank 1 local}\\ \text{systems on } X\end{array}\right\}.$$

rank 1 loc. system $\mathcal{F}$ s.t.

$m^*\mathcal{F} \cong p_1^*\mathcal{F} \otimes p_2^*\mathcal{F}$

$m: \mathrm{Pic}\times\mathrm{Pic} \to \mathrm{Pic}$.

$$\mathrm{Sym}^d X \xrightarrow{AJ_d} \mathrm{Pic}^d_X$$

$\mathcal{F}$ rank 1 loc. system on X.

$\mathrm{Sym}^d\mathcal{F}$ on $\mathrm{Sym}^d X$ again a rank 1 loc. system.

If $d > 2g-2$, AJ_d is a $\mathbb{P}^{d-g}$-bundle, so simply connected.

So, $\mathcal{F}^{(d)}$ descends to Pic^d_X for $d \gg 0$.

Check it has char. sheaf property.

Fargues: try to do the same for Local Class Field Theory.

<u>$d \geq 1$</u>. $\mathrm{Div}^d : \mathrm{Perf}_{\mathbb{F}_p} \ni S \longmapsto \{ (\mathcal{L}, v) : \mathcal{L}$ a line bundle of deg d, v is a non-zero section fiberwise on $S \}$.

(Analogue of $\mathrm{Sym}^d X$.)

Two useful properties.

(1) $\mathrm{Div}^d = [\, BC(\mathcal{O}(d)) \setminus \{0\} / E^\times]$
Banach-Colmez space.

$BC(\mathcal{O}(d)) : S \longmapsto H^0(X_S, \mathcal{O}(d))$

obvious map.

$\mathrm{Pic}^d = [\, * / E^\times]$

(2) - $\mathrm{Div}^1 = \mathrm{Spa}(\hat{E}^{un})^\diamond / \varphi^\diamond$
$\not\cong$
X_E

So, l-adic loc. systems on Div^1
$\updownarrow$
l-adic rep of W_E.

Frobenius decomplets I guess.

- $\mathrm{Div}^d = (\mathrm{Div}^1)^d / S_d$.

Now geometric proof of LCFT.

Take ρ character of W_E $\longleftrightarrow$ ℓ-adic loc. system $\mathcal{F}$ on Div^1.

Want: $\mathcal{F}$ descends to Pic^1 along $\mathrm{Div}^1 \to \mathrm{Pic}^1 = [*/E^\times]$.

Take $\mathcal{F}^{(d)}$ on Div^d. Must descend to Pic^d.

By point (1), enough to prove following.

Thm (Fargues). $BC(\mathcal{O}(d)\setminus\{0\})$ is simply connected (as a diamond), $d\geq 3$.

proof when $E = \overline{\mathbb{F}}_q((\pi))$. In this case,

$$BC(\mathcal{O}(d)\setminus\{0\})$$

$$\mathcal{O}(Y_{S,E}) = \Big\{ \sum x_n \pi^n : \|x_n\|\rho^n \to 0,\ \underset{0<\rho<1}{} \Big\}$$

$S = \mathrm{Spa}(A,A^+)$. $\qquad x_n \in A$.

$$BC(\mathcal{O}(d))(S) = H^0(X_{S,E}, \mathcal{O}(d)) = \mathcal{O}(Y_{S,E})^{\varphi=\pi^d}$$

$$= (A^{\circ\circ})^d.$$

$$\sum_{i=0}^{d-1} \sum_{k\in\mathbb{Z}} x_i^{q^{-k}} \pi^{i+kd} \longleftarrow (x_0,\dots,x_{d-1})$$

$$BC(\mathcal{O}(d)) = \mathrm{Spa}\, \overline{\mathbb{F}}_q[[x_0^{1/p^\infty},\dots,x_{d-1}^{1/p^\infty}]].$$

$$BC(\mathcal{O}(d))\setminus\{0\} = \mathrm{Spa}\, \overline{\mathbb{F}}_q[[x_0^{1/p^\infty},\dots,x_{d-1}^{1/p^\infty}]] \setminus V(x_0,\dots,x_{d-1}).$$

Zariski–Nagata.

$\underline{d\geq 2.}$ $\quad \mathrm{\acute{E}t}_{BC(\mathcal{O}(d))\setminus\{0\}} = \mathrm{\acute{E}t}_{\mathrm{Spa}\,\overline{\mathbb{F}}_q[[\ ]]\setminus V(x_0,\dots,x_{d-1})} \cong \mathrm{\acute{E}t}_{\mathrm{Spec}\,\overline{\mathbb{F}}_q[[-]]\setminus V(-)} \underset{d\geq 2}{=} \mathrm{\acute{E}t}_{\mathrm{Spec}\,\overline{\mathbb{F}}_q[[\dots]]} = \mathrm{pt}.$